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A boundary element method for anisotropic inhomogeneous elasticity

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Abstract

This paper is concerned with obtaining boundary integral equations for the numerical solution of the partial differential equations governing static deformations of inhomogeneous anisotropic elastic materials. The elastic parameters for the inhomogeneous materials are assumed to vary continuously with the spatial variables. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Boundary element; Boundary value problem; Inhomogeneous elasticity

1. Introduction

The study of boundary value problems for inhomogeneous materials has received considerable attention in recent years. This interest is partly related to the extensive use of composite materials in various engineering applications. In this connection many of the studies have been concerned with materials which are made up of two or more homogeneous parts and many problems have now been solved for materials of this type (see e.g. Clements, 1971; England, 1965). In comparison problems for materials in which the elastic parameters vary continuously with the spatial coordinates have received less attention. To some extent this is due to the inherent difficulties in solving boundary value problems for materials of this type. However in recent years some progress has been made with the analytical solution of particular problems for a restricted class of inhomogeneous materials (see e.g. Ang and Clements, 1987; Dhaliwal and Singh, 1978; Erdogan and Ozturk, 1992; Gibson, 1967; Gibson et al., 1971; Varley and Seymour, 1988).

The current study is concerned with the solution of boundary value problems for static deformations of inhomogeneous elastic materials. The elastic moduli vary continuously with the three Cartesian coordinates. A boundary integral formulation is used to obtain a solution to the governing differential equations and this is then applied to some particular boundary value problems.

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2. Statement of the problem and basic equations

Referred to a Cartesian frame $Ox_1x_2x_3$ consider an inhomogeneous anisotropic elastic material occupying a region Ω in R^3 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed surfaces. The equilibrium equations governing small deformations of this material may be written in the form

$$\frac{\partial}{\partial x_j}\left[c_{ijkl}(\mathbf{x})\,\frac{\partial u_k(\mathbf{x})}{\partial x_l}\right] = 0, \tag{2.1}$$

where $i,j,k,l = 1,2,3$, $\mathbf{x} = (x_1, x_2, x_3)$, u_k denotes the displacement, $c_{ijkl}(\mathbf{x})$ the elastic parameters and the repeated summation convention (summing from 1 to 3) is used for repeated Latin suffices. The stress displacement relations are given by

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}\,\frac{\partial u_k}{\partial x_l} \tag{2.2}$$

and the traction vector P_i on the boundary $\partial\Omega$ is defined as

$$P_i(\mathbf{x}) = \sigma_{ij}n_j = c_{ijkl}\,\frac{\partial u_k}{\partial x_l}\,n_j, \tag{2.3}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ denotes the outward pointing normal to the boundary $\partial\Omega$.

For all points in Ω the coefficients $c_{ijkl}(\mathbf{x})$ are required to satisfy the usual symmetry condition

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij} \tag{2.4}$$

and also sufficient conditions for the strain energy density to be positive. This requirement ensures that the system of partial differential equations is elliptic throughout Ω.

On the boundary $\partial\Omega$ the displacement u_k is specified on $\partial\Omega_1$ and the traction P_i is specified on $\partial\Omega_2$ where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. A solution to Eq. (2.1) is sought which is valid in Ω and satisfies the specified displacement on $\partial\Omega_1$ and traction on $\partial\Omega_2$.

3. Reduction to a linear constant coefficients equation

The coefficients in Eq. (2.1) are required to take the form

$$c_{ijkl}(\mathbf{x}) = c^{(0)}_{ijkl}\,g(\mathbf{x}), \tag{3.1}$$

where the $c^{(0)}_{ijkl}$ are constants. Also in addition to the symmetry condition (2.4) the $c^{(0)}_{ijkl}$ are required to satisfy the additional condition

$$c^{(0)}_{ijkl} = c^{(0)}_{ilkj}. \tag{3.2}$$

Eq. (2.1) may now be written in the form

$$c^{(0)}_{ijkl}\,\frac{\partial}{\partial x_j}\left(g\,\frac{\partial u_k}{\partial x_l}\right) = 0. \tag{3.3}$$

Consider the transformation

$$u_k = g^{-1/2}\,\psi_k. \tag{3.4}$$

Use of Eq. (3.4) in Eq. (3.3) provides the equation

$$g^{1/2}c^{(0)}_{ijkl}\frac{\partial^2\psi_k}{\partial x_j\partial x_l}+c^{(0)}_{ijkl}\frac{\partial g^{1/2}}{\partial x_j}\frac{\partial\psi_k}{\partial x_l}-c^{(0)}_{ijkl}\frac{\partial g^{1/2}}{\partial x_l}\frac{\partial\psi_k}{\partial x_j}-\psi_k c^{(0)}_{ijkl}\frac{\partial^2 g^{1/2}}{\partial x_j\partial x_l}=0, \tag{3.5}$$

where by virtue of Eq. (3.2) this equation reduces to

$$g^{1/2}c^{(0)}_{ijkl}\frac{\partial^2\psi_k}{\partial x_j\partial x_l}-\psi_k c^{(0)}_{ijkl}\frac{\partial^2 g^{1/2}}{\partial x_j\partial x_l}=0. \tag{3.6}$$

Thus if

$$c^{(0)}_{ijkl}\frac{\partial^2\psi_k}{\partial x_j\partial x_l}=0 \tag{3.7}$$

and

$$c^{(0)}_{ijkl}\frac{\partial^2 g^{1/2}}{\partial x_j\partial x_l}=0, \tag{3.8}$$

then Eq. (3.6) will be satisfied. Thus when g satisfies system (3.8) the transformation given by Eq. (3.4) transforms the linear system with variable coefficients (3.3) to the linear system with constant coefficients (3.7).

As a result of the symmetry property $c_{ijkl}=c_{klij}$ Eq. (3.8) consists of a system of six constant coefficients partial differential equations in the one dependent variable $g^{1/2}$. In general this system will be satisfied by a linear function of the three independent variables x_1, x_2, x_3. Thus $g(x)$ may be taken in the form

$$g(\mathbf{x})=(\alpha x_1+\beta x_2+\gamma x_3+\delta)^2, \tag{3.9}$$

where α, β, γ and δ are constants which may be used to fit the elastic parameters $c_{ijkl}(\mathbf{x})=c^{(0)}_{ijkl}g(\mathbf{x})$ to given numerical data.

Now substitution of Eqs. (3.1) and (3.4) into Eq. (2.3) yields

$$P_i=-P^{[g]}_{ik}\psi_k+P^{[\psi]}_i g^{1/2}, \tag{3.10}$$

where

$$P^{[g]}_{ik}(\mathbf{x})=c^{(0)}_{ijkl}\frac{\partial g^{1/2}}{\partial x_l}n_j, \tag{3.11}$$

$$P^{[\psi]}_{i}(\mathbf{x})=c^{(0)}_{ijkl}\frac{\partial \psi_k}{\partial x_l}n_j. \tag{3.12}$$

A boundary integral equation for the solution of Eq. (3.7) with ψ_i given on $\partial\Omega_1$ and $P^{[\psi]}_i$ given on $\partial\Omega_2$ may be written in the form (Clements, 1981)

$$\eta\psi_m(\mathbf{x}_0)=-\int_{\partial\Omega}\left[P^{[\psi]}_i(\mathbf{x})\Phi_{im}(\mathbf{x},\mathbf{x}_0)-\psi_i(\mathbf{x})\Gamma_{im}(\mathbf{x},\mathbf{x}_0)\right]\mathrm{d}s(\mathbf{x}), \tag{3.13}$$

for $m=1,2,3,$ where $\mathbf{x}_0=(a,b,c)$ is the source point, $\eta=0$ if $\mathbf{x}_0\notin\Omega$, $\eta=1$ if $\mathbf{x}_0\in\Omega$ and $\eta=1/2$ if $\mathbf{x}_0\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $\mathbf{x}_0$. The Φ_{im} in Eq. (3.13) is any solution of the equation

$$c^{(0)}_{ijkl}\frac{\partial^2\Phi_{im}(\mathbf{x},\mathbf{x}_0)}{\partial x_j\partial x_l}=\delta_{km}\delta(\mathbf{x}-\mathbf{x}_0) \tag{3.14}$$

and the Γ_{im} is given by

$$\Gamma_{im} = c^{(0)}_{ijkl}\frac{\partial \Phi_{km}}{\partial x_l}n_j. \tag{3.15}$$

For plane problems with $\mathbf{x}_0 = (a,b)$, $\mathbf{x} = (x_1,x_2)$, Φ_{im} and Γ_{im} are given by (see e.g. Clements and Jones, 1981)

$$\Phi_{im}(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\Re\left[\sum_{\alpha=1}^{2}A_{i\alpha}N_{\alpha k}\log(z_\alpha - c_\alpha)\right]d_{km}, \tag{3.16}$$

$$\Gamma_{im}(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\Re\left[\sum_{\alpha=1}^{2}L_{ij\alpha}N_{\alpha k}(z_\alpha - c_\alpha)^{-1}\right]n_j d_{km}, \tag{3.17}$$

where the indices now take the values 1 and 2, $\Re$ denotes the real part of a complex number, $z_\alpha = x_1 + \tau_\alpha x_2$ and $c_\alpha = a + \tau_\alpha b$, where τ_α are the two roots with positive imaginary part of the quartic in τ

$$|c^{(0)}_{i1k1} + c^{(0)}_{i2k1}\tau + c^{(0)}_{i1k2}\tau + c^{(0)}_{i2k2}\tau^2| = 0. \tag{3.18}$$

The $A_{i\alpha}$ occurring in Eq. (3.16) are the solutions of the system

$$\left(c^{(0)}_{i1k1} + c^{(0)}_{i2k1}\tau_\alpha + c^{(0)}_{i1k2}\tau_\alpha + c^{(0)}_{i2k2}\tau_\alpha^2\right)A_{k\alpha} = 0. \tag{3.19}$$

Also the $N_{\alpha k}$, $L_{ij\alpha}$ and d_{km} are defined by

$$\delta_{ik} = \sum_{\alpha=1}^{2}A_{i\alpha}N_{\alpha k}, \tag{3.20}$$

$$L_{ij\alpha} = \left(c^{(0)}_{ijk1} + \tau_\alpha c^{(0)}_{ijk2}\right)A_{k\alpha}, \tag{3.21}$$

$$\delta_{im} = -\frac{1}{2}\mathrm{i}\sum_{\alpha=1}^{2}\left\{L_{i2\alpha}N_{\alpha k} - \overline{L}_{i2\alpha}\overline{N}_{\alpha k}\right\}d_{km}, \tag{3.22}$$

where the bar denotes the complex conjugate and i denotes the square root of minus one.

For the three dimensional case Φ_{im} is given in Vogel and Rizzo (1973) in the form

$$\Phi_{im}(\mathbf{x},\mathbf{x}_0) = \frac{1}{8\pi^2|\mathbf{x}-\mathbf{x}_0|}\oint_{|\xi|=1}P_{ji}(\boldsymbol{\xi})\,\mathrm{d}s, \tag{3.23}$$

where ds is an element of arc length, and

$$P_{ij}(\boldsymbol{\xi}) = \frac{1/2\varepsilon_{imn}\varepsilon_{jrs}Q_{mr}(\boldsymbol{\xi})Q_{ns}(\boldsymbol{\xi})}{\det Q},$$

$$Q_{ik}(\boldsymbol{\xi}) = c^{(0)}_{ijkl}\xi_j\xi_l,$$

where ε_{imn} denotes the permutation symbol and $\det Q$ is the determinant of Q_{ik}. The integral in Eq. (3.23) is taken around a unit circle which has its center at the point $\mathbf{x}_0$ and lies in a plane perpendicular to the vector $\mathbf{x} - \mathbf{x}_0$.

Use of Eqs. (3.4) and (3.10) in Eq. (3.13) yields

$$\eta\, g^{1/2}(\mathbf{x}_0)\,u_m(\mathbf{x}_0) = -\int_{\partial\Omega}\Big\{P_i(\mathbf{x})[g^{-1/2}(\mathbf{x})\,\Phi_{im}(\mathbf{x},\mathbf{x}_0)] - u_i(\mathbf{x})[g^{1/2}(\mathbf{x})\,\Gamma_{im}(\mathbf{x},\mathbf{x}_0) - P^{[g]}_{ki}(\mathbf{x})\,\Phi_{km}(\mathbf{x},\mathbf{x}_0)]\Big\}\mathrm{d}s(\mathbf{x}). \tag{3.24}$$

This equation provides a boundary integral equation for determining u_m and P_m at all points of Ω.

4. A perturbation method

The analysis of Section 3 provides a boundary integral equation for the numerical solution of problems governed by Eq. (2.1) subject to the coefficients $c_{ijkl}(\mathbf{x})$ satisfying the conditions (3.1), (3.2) and (3.8). Within these constraints there is flexibility in the choice of the parameters in Eq. (3.9). Values of these parameters may be chosen to fit the $c_{ijkl}(\mathbf{x})$ to given numerical data within a restricted domain Ω. However despite this flexibility in the choice of the coefficients the fact remains that the analysis of Section 3 is only applicable for a somewhat restricted class of inhomogeneous materials.

In this section a boundary element procedure is obtained for a more general class of elastic coefficients $c_{ijkl}(\mathbf{x})$. In particular a boundary element procedure is derived for the case when the coefficients $c_{ijkl}(\mathbf{x})$ are perturbed about the forms given by Eq. (3.1). Specifically the coefficients $c_{ijkl}(\mathbf{x})$ are required to take the form

$$c_{ijkl}(\mathbf{x}) = c_{ijkl}^{(0)} g(\mathbf{x}) + \epsilon c_{ijkl}^{(1)}(\mathbf{x}), \tag{4.1}$$

where ϵ is a small parameter, $c_{ijkl}^{(1)}$ is a differentiable function, $c_{ijkl}^{(0)}$ satisfies Eq. (3.2) and $g(\mathbf{x})$ satisfies Eq. (3.8).

Substitution of Eq. (4.1) into Eq. (2.1) and use of the transformation Eq. (3.4) gives

$$c_{ijkl}^{(0)} \frac{\partial}{\partial x_j}\left[g \frac{\partial}{\partial x_l}\left(g^{-1/2}\psi_k\right)\right] = -\epsilon \frac{\partial}{\partial x_j}\left[c_{ijkl}^{(1)} \frac{\partial}{\partial x_l}\left(g^{-1/2}\psi_k\right)\right]. \tag{4.2}$$

Use of the analysis of Section 3 now permits this equation to be simplified to the form

$$c_{ijkl}^{(0)} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} = -\epsilon g^{-1/2}\left[\frac{\partial A_{ijk}}{\partial x_j}\psi_k + \left(A_{ijk} + \frac{\partial B_{ilkj}}{\partial x_l}\right)\frac{\partial \psi_k}{\partial x_j} + B_{ijkl}\frac{\partial^2 \psi_k}{\partial x_j \partial x_l}\right], \tag{4.3}$$

where

$$A_{ijk}(\mathbf{x}) = c_{ijkl}^{(1)} \frac{\partial g^{-1/2}}{\partial x_l} \quad \text{and} \quad B_{ijkl}(\mathbf{x}) = c_{ijkl}^{(1)} g^{-1/2}.$$

A solution to Eq. (4.3) is sought in the form

$$\psi_k(\mathbf{x}) = \sum_{r=0}^{\infty} \epsilon^r \psi_k^{(r)}(\mathbf{x}). \tag{4.4}$$

Substitution of Eq. (4.4) into Eq. (4.3) and equating coefficients of powers of ϵ yields

$$c_{ijkl}^{(0)} \frac{\partial^2 \psi_k^{(r)}}{\partial x_j \partial x_l} = h_i^{(r)}, \quad \text{for } r = 0, 1, \ldots, \tag{4.5}$$

where

$$h_i^{(0)}(\mathbf{x}) = 0, \tag{4.6}$$

$$h_i^{(r)}(\mathbf{x}) = -g^{-1/2}\left[\frac{\partial A_{ijk}}{\partial x_j}\psi_k^{(r-1)} + \left(A_{ijk} + \frac{\partial B_{ilkj}}{\partial x_l}\right)\frac{\partial \psi_k^{(r-1)}}{\partial x_j} + B_{ijkl}\frac{\partial^2 \psi_k^{(r-1)}}{\partial x_j \partial x_l}\right], \quad \text{for } r = 1, 2, \ldots \tag{4.7}$$

The boundary integral equation for Eq. (4.5) is

$$\eta \psi_m^{(r)}(\mathbf{x}_0) = -\int_{\partial\Omega}\left[P_i^{[\psi^{(r)}]}(\mathbf{x})\,\Phi_{im}(\mathbf{x},\mathbf{x}_0) - \psi_i^{(r)}(\mathbf{x})\,\Gamma_{im}(\mathbf{x},\mathbf{x}_0)\right]\mathrm{d}s(\mathbf{x}) + \int_{\Omega} h_i^{(r)}(\mathbf{x})\,\Phi_{im}(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}s(\mathbf{x}), \tag{4.8}$$

where $P_i^{[\psi^{(r)}]} = c_{ijkl}^{(0)}\left(\partial\psi_k^{(r)}/\partial x_l\right) n_j$ for $r = 0, 1, \ldots$

From Eqs. (3.4) and (4.4) the displacement u_k may be written in the series form

$$u_k(\mathbf{x}) = \sum_{r=0}^{\infty} \epsilon^r u_k^{(r)}(\mathbf{x}), \tag{4.9}$$

where $u_k^{(r)}$ corresponds to $\psi_k^{(r)}$ according to the relationship

$$\psi_k^{(r)} = g^{1/2} u_k^{(r)}, \quad \text{for } r = 0, 1, \ldots \tag{4.10}$$

Also

$$P_i^{[\psi^{(r)}]} = g^{1/2} P_i^{(r)} + u_k^{(r)} P_{ik}^{[g]}, \quad \text{for } r = 0, 1, \ldots, \tag{4.11}$$

where

$$P_i^{(r)} = c_{ijkl}^{(0)} \left(\partial u_k^{(r)} / \partial x_l \right) n_j, \quad \text{for } r = 0, 1, \ldots \tag{4.12}$$

Thus the integral Eq. (4.8) may be written in the form

$$\begin{aligned} \eta\, g^{1/2}(\mathbf{x}_0)\, u_m^{(r)}(\mathbf{x}_0) = &- \int_{\partial\Omega} \Big\{ P_i^{(r)}(\mathbf{x}) \left[g^{1/2}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0) \right] \\ &- u_i^{(r)}(\mathbf{x}) [g^{1/2}(\mathbf{x})\, \Gamma_{im}(\mathbf{x}, \mathbf{x}_0) - P_{ki}^{[g]}(\mathbf{x})\, \Phi_{km}(\mathbf{x}, \mathbf{x}_0)] \Big\} \mathrm{d}s(\mathbf{x}) \\ &+ \int_{\Omega} h_i^{(r)}(\mathbf{x})\, \Phi_{im}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \quad \text{for } r = 0, 1, \ldots \end{aligned} \tag{4.13}$$

Also, the function $h_i^{(r)}$ may be written in the form

$$\begin{aligned} h_i^{(r)} = -g^{-1/2} \Bigg\{ &\left[\frac{\partial}{\partial x_j} \left(g^{1/2} A_{ijk} \right) + \frac{\partial B_{ilkj}}{\partial x_l} \frac{\partial g^{1/2}}{\partial x_j} \right] u_k^{(r-1)} + \left[g^{1/2} A_{ijk} + \frac{\partial}{\partial x_l} \left(g^{1/2} B_{ilkj} \right) + B_{ijkl} \frac{\partial g^{1/2}}{\partial x_l} \right] \frac{\partial u_k^{(r-1)}}{\partial x_j} \\ &+ \left(g^{1/2} B_{ijkl} \right) \frac{\partial^2 u_k^{(r-1)}}{\partial x_j \partial x_l} \Bigg\} \quad \text{for } r = 1, 2, \ldots \end{aligned} \tag{4.14}$$

The corresponding value of P_i is given by

$$\begin{aligned} P_i &= c_{ijkl} \frac{\partial u_k}{\partial x_l} n_j = \left[c_{ijkl}^{(0)} g + \epsilon\, c_{ijkl}^{(1)} \right] \frac{\partial u_k}{\partial x_l} n_j = g\, c_{ijkl}^{(0)} \frac{\partial u_k}{\partial x_l} n_j + \epsilon\, c_{ijkl}^{(1)} \frac{\partial u_k}{\partial x_l} n_j \\ &= g\, c_{ijkl}^{(0)} \frac{\partial u_k^{(0)}}{\partial x_l} n_j + \sum_{r=1}^{\infty} \epsilon^r \left[g\, c_{ijkl}^{(0)} \frac{\partial u_k^{(r)}}{\partial x_l} + c_{ijkl}^{(1)} \frac{\partial u_k^{(r-1)}}{\partial x_l} \right] n_j = g P_i^{(0)} + \sum_{r=1}^{\infty} \epsilon^r \left[g P_i^{(r)} + G_i^{(r)} \right], \end{aligned} \tag{4.15}$$

where

$$G_i^{(r)}(\mathbf{x}) = c_{ijkl}^{(1)} \frac{\partial u_k^{(r-1)}}{\partial x_l} n_j. \tag{4.16}$$

To satisfy the boundary conditions in Section 2 it is required that $u_i^{(0)} = u_i$ on $\partial\Omega_1$ where u_i takes on its specified value on $\partial\Omega_1$. Also it is required that on $\partial\Omega_2$ $P_i^{(0)} = g^{-1} P_i$ where P_i takes on its specified value on $\partial\Omega_2$. It then follows from Eqs. (4.9) and (4.15) that for $r = 1, 2, \ldots$, $u_i^{(r)} = 0$ on $\partial\Omega_1$ and $P_i^{(r)} = -g^{-1} G_i^{(r)}$ on $\partial\Omega_2$.

The integral Eq. (4.13) may be used to find the numerical values of the unknown $u_i^{(r)}$ or $P_i^{(r)}$ on the boundary $\partial\Omega$ and the numerical values of $u_i^{(r)}$ and its derivatives in the domain Ω for $r = 0, 1, \ldots$ The u_i and P_i may then be obtained, respectively, from Eqs. (4.9) and (4.15) and the stresses σ_{ij} from Eq. (2.2).

5. Numerical results

To illustrate the application of the integral equations derived in the previous sections some plane strain boundary value problems for inhomogeneous materials are considered in this section.

The first problem involves the simple extension of a constrained inhomogeneous transversely isotropic slab. An analytical solution exists for the particular problem considered and this facilitates an assessment of the accuracy of the boundary element procedure.

The second problem concerns an inhomogeneous material consisting of layers of homogeneous transversely isotropic materials with the elastic constants varying from layer to layer. The particular problem considered facilitates an assessment of the effectiveness of the boundary element procedure for the solution of problems involving layered anisotropic materials.

The third problem concerns strip loading of an inhomogeneous isotropic halfspace and an isotropic layer of constant finite width. Problems of this type are important in geotechnical assessments involving loading of the earth's surface in cases where the material immediately below the surface is found to vary with depth (see e.g. Ward et al., 1968). The examples considered illustrate the suitability of the boundary element method for the solution of boundary value problem of this type (involving unbounded domains) and also demonstrate the applicability of the method for solving problems for isotropic materials as a limiting case of the anisotropic analysis.

Before proceeding to the specific problems it is appropriate at this point to relate the constants c_{ijkl} and the matrices $A_{k\alpha}$ and $L_{i2\alpha}$ of Section 3 to the specific constants relevant for plane strain problems for transversely isotropic and isotropic materials. For these problems the displacements u_k are taken in the form $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$ and $u_3 = 0$.

For transversely isotropic materials with the x_1-axis being normal to the transverse plane the non-zero elastic moduli $c_{ijkl}(\mathbf{x})$ of interest are $c_{1111}, c_{1122}, c_{2222}$ and c_{1212}. Let

$$c^{(0)}_{1111} = C, \quad c^{(0)}_{1122} = F, \quad c^{(0)}_{2222} = A, \quad c^{(0)}_{1212} = L, \tag{5.1}$$

where A, C, F and L are constants.

For plane problems for this class of materials Eq. (3.7) becomes

$$C\frac{\partial^2\psi_1}{\partial x_1^2} + L\frac{\partial^2\psi_1}{\partial x_2^2} + (F+L)\frac{\partial^2\psi_2}{\partial x_1 \partial x_2} = 0, \tag{5.2}$$

$$(L+F)\frac{\partial^2\psi_1}{\partial x_1 \partial x_2} + L\frac{\partial^2\psi_2}{\partial x_1^2} + A\frac{\partial^2\psi_2}{\partial x_2^2} = 0 \tag{5.3}$$

and the quartic equation (3.18) may be written

$$AL\tau^4 - (F^2 + 2FL - CA)\tau^2 + CL = 0 \tag{5.4}$$

so that in this case explicit analytical expressions for the roots may be obtained by treating Eq. (5.4) as a quadratic in τ^2.

If τ_1^2 and τ_2^2 are the roots of Eq. (5.4) then from Eq. (3.19) it follows that the matrix $A_{k\alpha}$ is given by

$$A_{k\alpha} = \begin{bmatrix} -\frac{A\tau_1^2+L}{\tau_1(F+L)} & -\frac{A\tau_2^2+L}{\tau_2(F+L)} \\ 1 & 1 \end{bmatrix}$$

and hence from Eq. (3.21) the matrix $L_{i2\alpha}$ takes the form

$$L_{i2\alpha} = \begin{bmatrix} L\left(\frac{F-A\tau_1^2}{F+L}\right) & L\left(\frac{F-A\tau_2^2}{F+L}\right) \\ -L\left(\frac{F-A\tau_1^2}{\tau_1(F+L)}\right) & -L\left(\frac{F-A\tau_2^2}{\tau_2(F+L)}\right) \end{bmatrix}.$$

The matrices $N_{k\alpha}$ and d_{km} defined, respectively, by Eqs. (3.20) and (3.22) may be readily calculated by employing the above two matrices.

In the particular case of isotropic materials the constants c_{1111}, c_{2222}, c_{1212} and c_{1122} are given by

$$\begin{aligned} c_{1111} &= \tfrac{3E}{1+v} + \eta, \\ c_{2222} &= \tfrac{3E}{1+v}, \\ c_{1212} &= c_{1122} = \tfrac{E}{2(1+v)}, \end{aligned} \tag{5.5}$$

where E is Young's modulus, v is Poisson's ratio and η is a small constant which facilitates the inclusion of isotropy as a limiting case (as $\eta \to 0$) of the anisotropic analysis.

For general anisotropy (or for transverse isotropy when the axes are not aligned with the symmetry planes) the roots of the quartic polynomial (3.18) must be obtained numerically and then the homogeneous linear algebraic system of Eq. (3.19) readily yields the $A_{k\alpha}$. The $L_{i2\alpha}$ are obtained from Eq. (3.21) by directly substituting the known τ_α and $A_{k\alpha}$.

In implementing the boundary element method (BEM) to solve the subsequent problems the boundary $\partial\Omega$ in Eq. (4.13) is discretised into a number of segments of equal length and the unknowns assumed to be constant on each segment (Clements and Jones, 1981). Also where domain integrals are required the domain Ω in Eq. (4.13) is divided into a number of equal squares with the integrands calculated at the midpoints and assumed to be constant over each of the squares. The number of squares is increased to ensure the required level of accuracy in the domain integral. In the cases when domain integrals are required in the following problems it is sufficient to divide the domain into 25 equal squares in order to achieve convergence to the numerical values given in the relevant tables.

5.1. Extension of a constrained slab

Consider the boundary value problem given in Fig. 1 for a material with elastic coefficients

$$c'_{ijkl} = c_{ijkl}/\widehat{c}, \tag{5.6}$$

where $\widehat{c}$ is a reference elastic modulus and

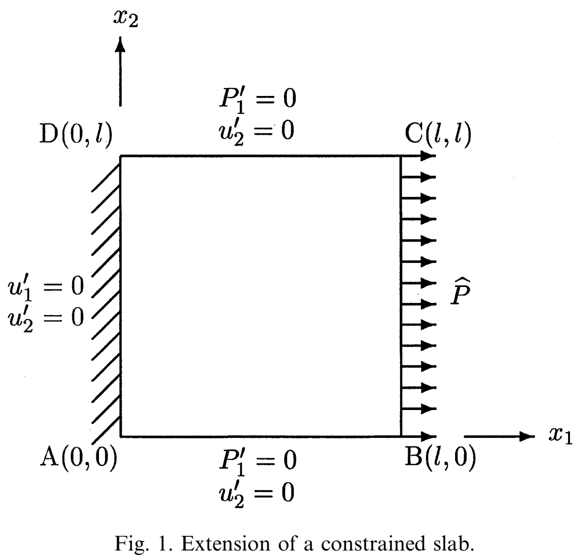

Fig. 1. Extension of a constrained slab.

$$\begin{aligned} c'_{1111} &= 6.14(1+x'_1)^2, \qquad c'_{1122} = 2.14(1+x'_1)^2, \\ c'_{2222} &= 5.96(1+x'_1)^2, \qquad c'_{1212} = 1.64(1+x'_1)^2 \end{aligned} \tag{5.7}$$

with $x'_i = x_i/l$ (for $i = 1, 2$) where l is a reference length.

The numerical values 6.14, 2.14, 5.96 and 1.64 in Eq. (5.7) when multiplied by 10^{11} dynes/cm^2 are the elastic constants for a particular homogeneous transversely isotropic material (Clements, 1981) and are chosen here purely for illustrative purposes. The elastic moduli (5.7) take the form (4.1) with

$$\begin{aligned} g &= (1+x'_1)^2, & \epsilon &= 0.25, \\ c'^{(0)}_{1111} &= 6.14, & c'^{(0)}_{1122} &= 1.89, \\ c'^{(0)}_{2222} &= 5.96, & c'^{(0)}_{1212} &= 1.89, \\ c'^{(1)}_{1111} &= 0, & c'^{(1)}_{1122} &= (1+x'_1)^2, \\ c'^{(1)}_{2222} &= 0, & c'^{(1)}_{1212} &= -(1+x'_1)^2 \end{aligned}$$

where $c'^{(m)}_{ijkl} = c^{(m)}_{ijkl}/\widehat{c}$ for $m = 0, 1$. Referred to Fig. 1 the boundary conditions are

$$\begin{aligned} P'_1 &= 0, \quad u'_2 = 0, \qquad \text{on AB}, \\ P'_1 &= 1, \quad P'_2 = 0, \qquad \text{on BC}, \\ P'_1 &= 0, \quad u'_2 = 0, \qquad \text{on CD}, \\ u'_1 &= 0, \quad u'_2 = 0, \qquad \text{on AD}, \end{aligned}$$

where $u'_i = u_i/\widehat{u}$, $P'_i = P_i/\widehat{P}$ with $\widehat{u}$ a reference displacement and $\widehat{P} = \widehat{c}\,\widehat{u}/l$ is a reference traction.

This problem admits the analytical solution

$$u'_1 = x'_1/[c'^{(0)}_{1111}(1+x'_1)], \quad u'_2 = 0$$

with the stress given by

$$\sigma'_{11} = 1, \quad \sigma'_{12} = 0, \quad \sigma'_{22} = c^{(0)}_{1122}/c^{(0)}_{1111},$$

where $\sigma'_{ij} = \sigma_{ij}/\widehat{P}$.

Tables 1 and 2 provide the analytical and BEM results for the non-zero displacement and stresses for some sample points in the domain Ω for the cases when the boundary $\partial\Omega$ is divided into 80, 160 and 320 segments for the displacement and 160 and 320 segments for the stresses.

The results indicate the convergence of the numerical solution to the exact values obtained from the analytical solution as the number of boundary segments increases.

5.2. Deformation of a layered anisotropic elastic slab

The analysis of Sections 2–4 was obtained under the assumption that the elastic coefficients $c_{ijkl}(\mathbf{x})$ satisfy sufficient differentiability conditions in Ω. However, in applications, the methods derived may be

Table 1
Displacements for the constrained slab

Position (x'_1, x'_2)	80 segments u'_1	160 segments u'_1	320 segments u'_1	Analytical u'_1
(0.1,0.5)	0.0142	0.0145	0.0147	0.0148
(0.3,0.5)	0.0368	0.0372	0.0374	0.0376
(0.5,0.5)	0.0534	0.0538	0.0540	0.0543
(0.7,0.5)	0.0661	0.0666	0.0668	0.0671
(0.9,0.5)	0.0762	0.0766	0.0769	0.0771

Table 2
Stresses for the constrained slab

Position	160 segments		320 segments		Analytical	
(x_1', x_2')	σ_{11}'	σ_{22}'	σ_{11}'	σ_{22}'	σ_{11}'	σ_{22}'
(0.1,0.5)	0.9973	0.3484	0.9979	0.3484	1.0000	0.3485
(0.3,0.5)	0.9969	0.3499	0.9980	0.3490	1.0000	0.3485
(0.5,0.5)	0.9974	0.3502	0.9982	0.3491	1.0000	0.3485
(0.7,0.5)	0.9981	0.3497	0.9986	0.3488	1.0000	0.3485
(0.9,0.5)	0.9992	0.3488	0.9992	0.3482	1.0000	0.3485

used for both the case when the coefficients satisfy these conditions and also as approximate methods for the solution of problems for a wide class of materials consisting of discrete layers of homogeneous anisotropic materials. For layered materials in which the material parameters do not vary too greatly from layer to layer, the approximate solution obtained using this method may be used to obtain accurate numerical values for the displacement and stress fields.

To illustrate the application of the integral equations of the previous section to problems involving layered media consider the problem of the compression of a layered slab in frictionless contact with a rigid base along one side (see Fig. 2).

The material consists of 10 homogeneous anisotropic layers which lie in the intervals

$$0.1n \leqslant x_1' \leqslant 0.1(n+1) \quad \text{for } n = 0, 1, 2, \ldots, 9.$$

The constant elastic moduli in each of these layers are given by

$$\begin{aligned} c_{1111}' &= 6.14[1 + \epsilon \sin(0.1n\pi)], \\ c_{1122}' &= 1.64[1 + \epsilon \sin(0.1n\pi)], \\ c_{2222}' &= 5.96[1 + \epsilon \sin(0.1n\pi)], \\ c_{1212}' &= 1.64[1 + \epsilon \sin(0.1n\pi)], \end{aligned} \tag{5.8}$$

for $n = 0, 1, 2, \ldots, 9$ with ϵ a small parameter. The boundary conditions for this problem are (see Fig. 2)

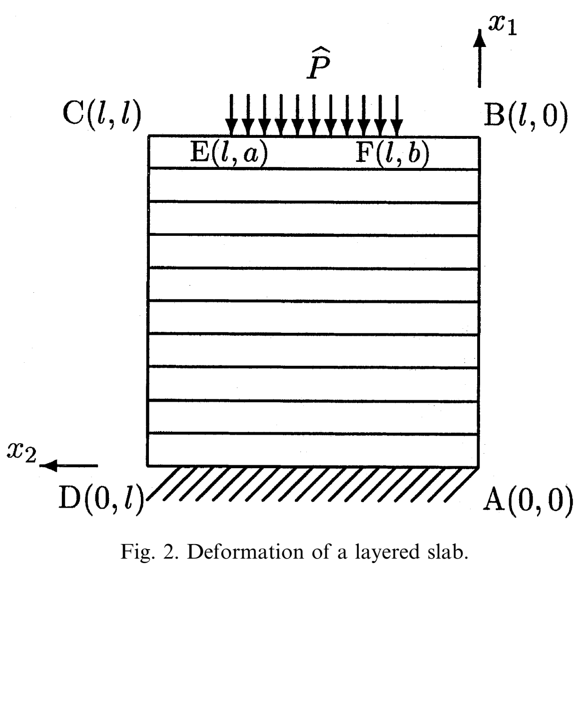

Fig. 2. Deformation of a layered slab.

$$
\begin{aligned}
&u_1 = 0, && P_2 = 0, && \text{on DA},\\
&P_1 = 0, && P_2 = 0, && \text{on AB, CD, CE and FB},\\
&P_1 = -\widehat{P}, && P_2 = 0, && \text{on EF}.
\end{aligned}
$$

Case (i) Loading along the whole side BC ($a = l$, $b = 0$): For the case when $a = l$ and $b = 0$ so that the loaded region on $x_1 = l$ extends along the whole side BC it is possible to obtain an analytical solution to this problem by taking displacements in each homogeneous layer in the form

$$
\begin{aligned}
u'_1 &= a_n x'_1 + b_n,\\
u'_2 &= c_n x'_2 + d_n,
\end{aligned}
\tag{5.9}
$$

for $0.1n \leqslant x'_1 \leqslant 0.1(n+1)$ and $n = 0, 1, 2, \ldots, 9$ where a_n, b_n, c_n, d_n are constants. Imposing continuity conditions on the displacements and tractions at each of interfaces $x'_1 = 0.1, 0.2, \ldots, 0.9$ together with the boundary conditions at the boundaries AB, BC, CD, and DA leads to 40 linear algebraic equations which may be solved for the 40 unknowns $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, \ldots, a_9, b_9, c_9, d_9$. Eq. (5.9) then provides the displacement at all points of the layered slab.

In order to use the BEM of Section 4 to obtain an approximate solution to this problem the discrete elastic moduli given by Eq. (5.8) are approximated by the continuous moduli

$$
\begin{aligned}
c'_{1111} &= 6.14[1 + \epsilon \sin(\pi x'_1)],\\
c'_{1122} &= 1.64[1 + \epsilon \sin(\pi x'_1)],\\
c'_{2222} &= 5.96[1 + \epsilon \sin(\pi x'_1)],\\
c'_{1212} &= 1.64[1 + \epsilon \sin(\pi x'_1)].
\end{aligned}
$$

These elastic moduli take form (4.1) with

$$
\begin{aligned}
&g = 1,\\
&c'^{(0)}_{1111} = 6.14, && c'^{(0)}_{1122} = 1.64,\\
&c'^{(0)}_{2222} = 5.96, && c'^{(0)}_{1212} = 1.64,\\
&c'^{(1)}_{1111} = 6.14 \sin(\pi x'_1), && c'^{(1)}_{1122} = 1.64 \sin(\pi x'_1),\\
&c'^{(1)}_{2222} = 5.96 \sin(\pi x'_1), && c'^{(1)}_{1212} = 1.64 \sin(\pi x'_1).
\end{aligned}
$$

Results for some selected points in the slab and selected values of ϵ are given in Tables 3 and 4. The BEM results in Tables 3 and 4 were obtained using 320 boundary segments. The difference between the values obtained using the BEM and the analytical solution decreases as the value of the parameter ϵ decreases. This is to be expected at least for two reasons; namely that the perturbation method in Section 4 will generally give a more accurate approximation as the parameter ϵ decreases, and secondly the variation in the values of the elastic moduli from layer to layer will decrease as ϵ becomes smaller so that the

Table 3
Displacement u'_1 for the layered slab

Position	$\epsilon = 0.1$		$\epsilon = 0.2$		$\epsilon = 0.3$	
(x'_1, x'_2)	BEM	Analytical	BEM	Analytical	BEM	Analytical
(0.25, 0.5)	−0.0421	−0.0429	−0.0405	−0.0419	−0.0388	−0.0411
(0.45, 0.5)	−0.0741	−0.0755	−0.0694	−0.0723	−0.0646	−0.0695
(0.65, 0.5)	−0.1059	−0.1075	−0.0976	−0.1017	−0.0894	−0.0967
(0.85, 0.5)	−0.1386	−0.1401	−0.1278	−0.1321	−0.1171	−0.1252

Table 4
Stresses σ'_{11} for the layered slab

Position	$\epsilon = 0.1$		$\epsilon = 0.2$		$\epsilon = 0.3$	
(x'_1, x'_2)	BEM	Analytical	BEM	Analytical	BEM	Analytical
(0.25, 0.5)	−0.9948	−1.0000	−0.9790	−1.0000	−0.9530	−1.0000
(0.45, 0.5)	−0.9885	−1.0000	−0.9559	−1.0000	−0.9031	−1.0000
(0.65, 0.5)	−0.9911	−1.0000	−0.9639	−1.0000	−0.9205	−1.0000
(0.85, 0.5)	−0.9986	−1.0000	−0.9914	−1.0000	−0.9799	−1.0000

approximation of the discrete material by an inhomogeneous material with continuous moduli will involve a smaller error.

Case (ii) Loading along part of side BC ($a < l$ and/or $b > 0$): For this case there is no simple analytical solution to the boundary value problem. The BEM may be used to obtain an approximate numerical solution and some results for the displacement $u_1(l, x_2)$ on the upper side of the slab are given in Fig. 3 for $\epsilon = 0.1$ and various loaded regions EF. As expected the normal displacement on BC decreases as the loaded region decreases and the extent to which this occurs is quantified in Fig. 3.

5.3. Strip loading of a half-space and a finite layer

In this section the analysis of the previous sections is used to consider strip loading of an isotropic inhomogeneous half-space (Fig. 4) and a layer of constant finite width (Fig. 5). Problems of this type have been considered by Gibson (1967) who considered an incompressible (Poisson's ratio 1/2) isotropic half-space with a shear modulus which varies linearly with depth and hence referred to the geometry in Fig. 4 assumes the form

$$\mu(x_2) = \mu(0) + mx_2 \tag{5.10}$$

for some constant m. Subsequently Gibson et al. (1971) considered a similar strip loading problem for the same class of materials with the half-space replaced by a finite layer. Gibson et al. (1971) observed that in a geotechnical assessment of a possible site for a large proton accelerator Ward et al. (1968) had found that the elastic properties of the site varied with depth. In their paper Ward et al. (1968) indicated that their analysis could be advantageously refined by an analysis in which the varying stiffness with depth was taken

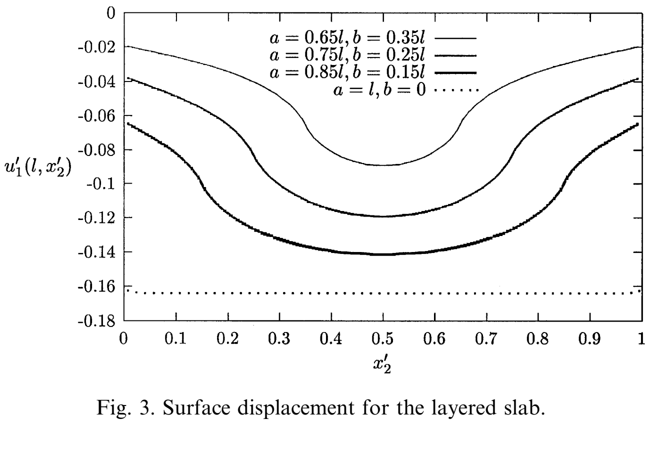

Fig. 3. Surface displacement for the layered slab.

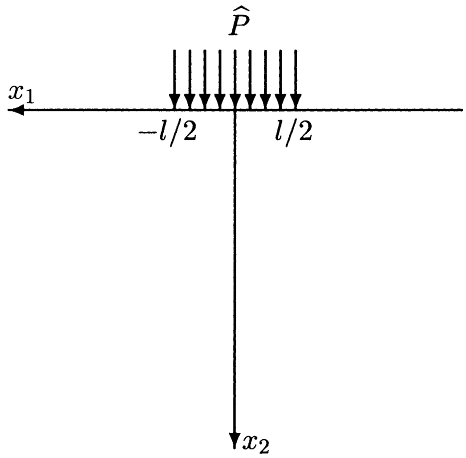

Fig. 4. Strip loading of a half-space.

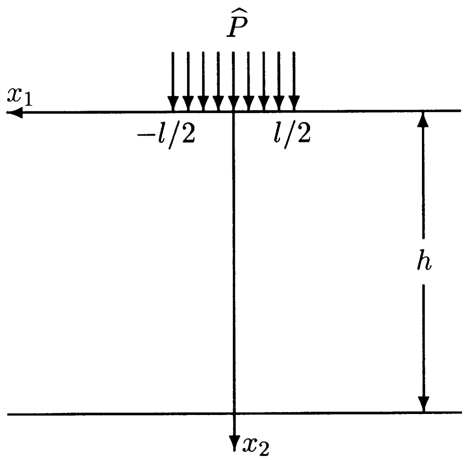

Fig. 5. Strip loading of a layer on a rough rigid base.

into account. Gibson et al. (1971) provide a possible means for such an analysis in the case when the material is incompressible with shear modulus of the linear form (5.10).

The analysis provided in this paper provides a possible means of investigating this type of problem for another class of isotropic inhomogeneous materials which may be closely associated with the numerical data given by Ward et al. (1968). In particular, the surface displacement due to strip loading of a half-space and finite layer is considered for the case when Poisson's ratio is 0.25 and the stiffness is given by

$$E = E_0(1 + \alpha x_2)^2,$$

where E_0 and α are constants. If x_2 is measured in metres then for the particular values of $E_0 = 2800\ \mathrm{kg\,cm^{-2}}$ and $\alpha = 0.12\ \mathrm{m^{-1}}$ this form for the stiffness E may be used to closely approximate numerical values given by Ward et al. (1968). From Eq. (5.5) the relevant non-zero constants c'_{ijkl} are taken in the form

$$
\begin{aligned}
c'_{1111} &= c_{1111}/E_0 = \left(\tfrac{3}{2(1+\nu)} + \widehat{\eta}\right)(1+\alpha' x'_2)^2, \\
c'_{2222} &= c_{2222}/E_0 = \tfrac{3}{2(1+\nu)}(1+\alpha' x'_2)^2, \\
c'_{1212} &= c_{1212}/E_0 = \tfrac{1}{2(1+\nu)}(1+\alpha' x'_2)^2, \\
c'_{1122} &= c_{1122}/E_0 = \tfrac{1}{2(1+\nu)}(1+\alpha' x'_2)^2,
\end{aligned}
\tag{5.11}
$$

where $\alpha' = \alpha l$, $x'_2 = x_2/l$ are non-dimensional quantities and $\widehat{\eta}$ is a non-dimensional small constant included so as to provide the isotropic limit (as $\widehat{\eta} \to 0$) of the anisotropic analysis. For the practical purposes of the numerical calculations reported in this section for isotropic materials $\widehat{\eta} = 0.0001$.

The analysis of Section 3 may be employed where the coefficients are taken in the form (3.1) with

$$
\begin{aligned}
&g = (1+\alpha' x'_2)^2, && c'^{(0)}_{1111} = \tfrac{3}{2(1+\nu)} + \widehat{\eta}, \\
&c'^{(0)}_{2222} = \tfrac{3}{2(1+\nu)}, && c'^{(0)}_{1212} = c'^{(0)}_{2222} = \tfrac{1}{2(1+\nu)}.
\end{aligned}
$$

For the half-space numerical values of the surface displacement $u'_2 = u_2/l$ on $x_2 = 0$ are given in Fig. 6 for $\alpha' = 0.12$, $\alpha' = 0.24$ and $\alpha' = 0.36$. The graph clearly indicates the reduction in the magnitude of the surface displacement u'_2 as the stiffness $E = E_0(1+\alpha' x'_2)^2$ increases with α'.

To obtain these results the integral in Eq. (3.24) was taken along the line $x_2 = 0$. For the practical purposes of obtaining numerical values the traction vector components were specified as $P_1 = 0, P_2/\widehat{P} = 1$ on the contact region $|x'_1| < 0.5$ and the interval divided into S_1 segments of equal length. In the two intervals defined by $0.5 < |x'_1| < k$ the traction-free condition $P_1 = 0, P_2 = 0$ was applied and each of the two intervals divided into S_2 segments of equal length. In the two intervals $k < |x'_1| < k + 5.5$ the displacement components u_1 and u_2 were both taken to be zero and each of the two intervals divided into S_3 segments of equal length. The number of segments $S_1 + 2S_2 + 2S_3$ and the value of k were then increased until the numerical values converged and the displacement as $|x'_1| \to k$ was zero to three decimal places. To satisfy these requirements for the numerical examples considered in this paper it was sufficient to take $S_1 = 40$, $S_2 = 244$, $S_3 = 22$ and $k = 16.5$.

To obtain some indication of the validity and accuracy of the results obtained using this method the values of the displacement were calculated for the homogeneous case $\alpha = 0$. In this case an analytical solution exists for the problem under consideration in the form (Clements, 1981)

$$
u_k = \sum_{\alpha=1}^{2} \left[A_{k\alpha} M_{\alpha j} \chi_j(z_\alpha) - \overline{A}_{k\alpha} \overline{M}_{\alpha j} \chi_j(\overline{z}_\alpha) \right], \tag{5.12}
$$

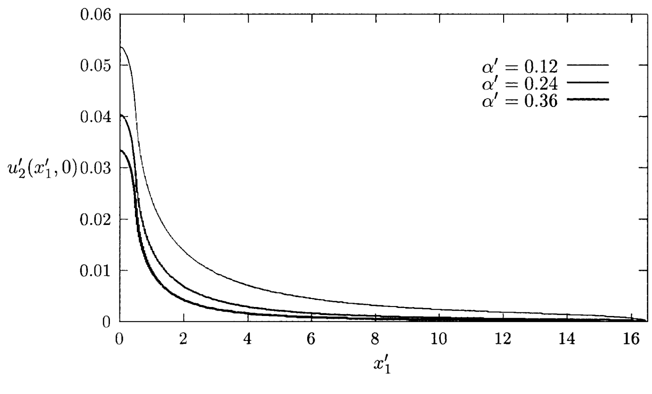

Fig. 6. Surface displacement for the half-space.

with

$$\chi_1(z_\alpha) = 0$$
$$\chi_2(z_\alpha) = \frac{\widehat{P}}{2\pi i}\{(z_\alpha - l/2)\log(z_\alpha - l/2) - (z_\alpha + l/2)\log(z_\alpha + l/2)\}$$

where the bar denotes the complex conjugate, $z_\alpha = x_1 + \tau_\alpha x_2$ and the matrix $M_{\alpha j}$ is the inverse of the matrix $L_{i2\alpha}$ so that

$$\sum_{\alpha=1} L_{i2\alpha} M_{\alpha j} = \delta_{ij}.$$

The magnitude of the displacement given by Eq. (5.12) tends to infinity as $|z_\alpha| \to \infty$. This is a well-known feature of solutions to half-space problems of this type and as a result of this difficulty such solutions can only be considered to give useful information on the displacement near to the contact region (Love, 1927). In order to use Eq. (5.12) to obtain meaningful data for the surface displacement for the specific problem considered here it is convenient to follow the procedure adopted by Poulos and Davis (1974) and consider the displacement of one point relative to another point with both points being in the vicinity of the contact region. Taking one point to be the normal surface displacement at the origin $u_2(0,0)$, the difference in displacement $u_2'(0,0) - u_2'(x_1,0)$, calculated from Eq. (5.12), is given in Table 5 together with numerical values of the same difference in displacement calculated using the BEM of Section 3 with the boundary and boundary conditions as described above. As can be seen from Table 5 the difference in displacements obtained from the analytical and boundary element solutions are in close agreement for the indicated points.

For $\alpha \neq 0$ no simple analytical solution is available for the half-space problem. The normal displacement $u_2'(x_1',0)$ is given graphically in Fig. 6 for the cases $\alpha' = 0.12$, $\alpha' = 0.24$ and $\alpha' = 0.36$. As expected the displacement decreases as the rigidity increases with increasing α' and the extent of this decrease is shown in Fig. 6.

For the strip loading of a layer of finite width on a rough rigid base the normal surface displacement $u_2'(x_1',0)$ is given in Figs. 7 and 8 for three values of α' and two layer widths $h = 2l$ and $h = 5l$. Again as expected the magnitude of the displacement decreases as α' increases and the extent to which this occurs is quantified in the figures for the values of α' indicated.

To obtain the numerical results for the strip the integral in Eq. (3.24) was taken along the boundary of the rectangle $|x_1'| \leqslant k$, $|x_2'| \leqslant h$. The side on $x_2 = 0$ was divided into S_1 equal segments over the contact region $|x_1'| < 0.5$ and S_2 equal segments on each of the two intervals specified by $0.5 < |x_1'| < k$. Also the side on $x_2 = h$ was divided into S_3 equal segments and the two sides $x_1' = \pm k$ divided into S_4 equal segments. The number of segments $S_1 + 2S_2 + S_3 + S_4$ and the value k were then increased until the numerical values converged and the contribution of the integral along $x_1' = \pm k$ was zero to three decimal places. To satisfy these requirements for the numerical results given in Figs. 7 and 8 it was sufficient to take $S_1 = 40$, $S_2 = 200$, $S_3 = 44$, $S_4 = 40$ and $k = 5.5$.

Table 5
$u_2'(0,0) - u_2'(x_1',0)$ for a homogeneous half-space

$(x_1',0)$	BEM	Analytical
$(0,0)$	0.0000	0.0000
$(1,0)$	0.0355	0.0351
$(2,0)$	0.0507	0.0506
$(3,0)$	0.0596	0.0594
$(4,0)$	0.0660	0.0656

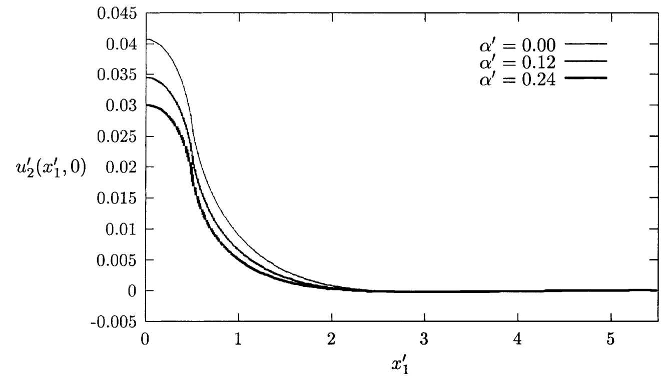

Fig. 7. Surface displacement for the layer of width $h = 2l$.

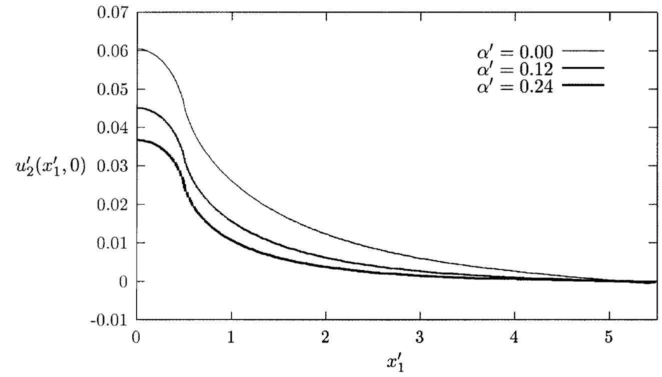

Fig. 8. Surface displacement for the layer of width $h = 5l$.

6. Summary

Some BEMs for the solution of certain classes of boundary value problems of elasticity for anisotropic inhomogeneous media have been derived. The methods are generally easy to implement to obtain numerical values for particular problems. They can be applied to a wide class of important practical problems for inhomogeneous anisotropic materials. The numerical results obtained using the methods to solve some sample problems indicate that they can provide accurate numerical solutions.

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